[7] Greenspan, H. P. and Benney, D. J., "On Shear Layer Instability, Breakdown and Transition," *Journal of Fluid Mechanics*, Vol. 15, 1963, p. 133.

[8] Emmons, H. W., "The Laminar-Turbulent Transition in a Boundary Layer," Pt. I, *Journal of Atmospheric Sciences*, Vol. 18, 1958, p. 490.

[9] Starr, V. P., *"Physics of Negative Viscosity Phenomena,"* McGraw-Hill, New York, 1968.

[10] Squire, H. B., "On the Stability of Three-Dimensional Disturbances of Viscous Flow between Parallel Walls," *Proceedings of the Royal Society*, Vol. A 142, 1933, p. 621.

[11] Kovasznay, L. S. G., "Detailed Flow Field in Transition," *Proceedings of the 1962 Heat Transfer and Fluid Mechanics Conference*, Stanford University Press, 1962.

[12] Kim, H. T., Kline, S. J., and Reynolds, W. C., "An Experimental Study of Turbulence Production near a Smooth Wall in a Turbulent Boundary Layer with Zero Pressure Gradient," Rept. MD 20, 1968, Dept. of Engineering, Stanford Univ., Calif.

[13] Coles, D., "The Law of the Wake and the Turbulent Boundary Layer," *Journal of Fluid Mechanics*, Vol. 1, 1956, p. 191.

[14] Klebanoff, P. S., "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," Rept. 1247, 1955, NACA.

[15] Laufer, J., "The Structure of Fully Developed Pipe Flow," Rept. 1174, 1954, NACA.

[16] Grant, H. L., Stewart, R. W., and Moilliet, A., "Turbulence Spectra from a Tidal Channel," *Journal of Fluid Mechanics*, Vol. 12, 1962, p. 241.

[17] Gupta, A. K., "An Experimental Investigation of the Viscous Sublayer Region in a Turbulent Boundary Layer," Ph.D. thesis, 1970, Univ. of Southern California.

[18] Willmarth, W. W. and Bo Jang Tu., "Structure of Turbulence in the Boundary Layer near the Wall," *The Physics of Fluids*, Supplement 10, 1967, p. 134.

[19] Landahl, M. T., "A Wave-Guide Model for Turbulent Shear Flow," *Journal of Fluid Mechanics*, Vol. 29, 1967, p. 331.

[20] Willmarth, W. W. and Woolridge, C. E., "Measurement of the fluctuating pressure at the wall beneath a thick boundary layer," *Journal of Fluid Mechanics*, Vol. 11, 1962, p. 187.

[21] Barger, W. R., Garrett, W. D., Mollo-Christensen, E. L., and Ruggles, K. W., "Effects of an Artificial Sea Slick upon the Atmosphere and the Ocean," *Journal of Applied Meteorology*, Vol. 9, 1970, pp. 396–400.

[22] Virk, P., "An Elastic Sublayer Model for Drag Reduction by Dilute Solutions of Linear Micromoleculae," to be published in the *Journal of Fluid Mechanics*, 1970.

[23] Lighthill, J. M., "On Sound Generated Aerodynamically," Pt. II, *Proceedings of the Royal Society*, Vol. A 222, 1954, p. 1.

JULY 1971 AIAA JOURNAL VOL. 9, NO. 7

Reduction of Launch Vehicle Injection Errors by Trajectory Shaping

RICHARD ROSENBAUM*
Lockheed Missiles & Space Company, Palo Alto, Calif.

A method is presented for shaping a booster trajectory to minimize the sensitivity of terminal constraints to variations in vehicle or atmosphere parameters. The unique feature of the problem is that the payoff function depends not only on the state variables and the controls, but also on the adjoint variables (Lagrange multipliers). In order to apply the gradient optimization technique, two new sets of differential equations must be integrated. An example, using the Scout booster, is given in which it is shown that the sensitivity of terminal altitude to variations in first stage burn rate can be reduced by 50%.

Nomenclature

f = vector of derivatives of state variables
P = vector of system variables
q = vector of system parameters
t = time
T = thrust
x = vector of state variables
α = vector of control variables
θ = thrust attitude angle
λ = vector of adjoint variables associated with the altitude constraint
λ_ψ = vector of adjoint variables associated with constraint ψ
Λ_P = influence coefficient relating P to ψ
Λ_φ = influence coefficient relating α to φ
μ = vector of adjoint variables associated with λ_ψ
ν = vector of adjoint variables associated with the sensitivity payoff
ξ = vector of control parameters
φ = sensitivity payoff

Superscripts

f = final
i = initial
T = transpose

Subscripts

n = nominal
p = perturbed

Presented as Paper 70-1078 at the AAS/AIAA Astrodynamics Conference, Santa Barbara, Calif., August 19–21, 1970; submitted September 15, 1970; revision received January 29, 1971. This work was supported by the Langley Research Center under Contract NAS 1-8920. The author would like to thank J. V. Breakwell of Stanford University for several illuminating discussions and R. E. Willwerth for assistance in performing the study. Z. Taulbee was responsible for the computer programing.

* Research Scientist, Mathematics and Operations Research Laboratory. Member AIAA.

Introduction

THE advent of the high-speed digital computer, together with the development of the gradient method of trajectory optimization, has made it possible to rapidly determine the maximum performance of booster vehicles. In many cases, however, the capability of the booster exceeds the mission requirements. The payload to be placed in orbit, for example, may weigh considerably less than the maximum payload that the booster can deliver into orbit. A variety of trajectories will satisfy the mission requirement. It is logical to inquire whether the excess booster capacity can be used to improve some characteristic of the trajectory.

In this paper, a method is presented for using the excess booster capability to shape the trajectory in order to reduce the sensitivity of the terminal constraints to variations in booster or atmosphere parameters. This is particularly important if an open-loop guidance system is being used because there is no way to correct the pitch program to compensate for nonstandard conditions.

Trajectory shaping to minimize sensitivity has been employed in Refs. 1 and 2. Leondes and Volgenau[1] reduce the impact error of a ballistic missile by finding a trajectory which minimizes a weighted sum of the partial derivatives of range with respect to the state variables at burnout. Their payoff function is dependent only on the final values of the state variables. A solution can therefore be found by using one of the standard methods for solving the two-point boundary value problem of the calculus of variations. They use the Newton-Raphson method, and report a significant reduction in sensitivity. Watson and Stubberud[2] attempt to reduce the effect of atmospheric density variations on the impact point of an entry vehicle. They develop a differential equation for the sensitivity of range to sea-level density and treat this sensitivity as a state variable. Their optimization scheme requires the guessing of adjoint variables. They reported difficulty in converging to the optimal solution and were not able to reduce the sensitivity significantly.

The gradient method is used here. Orbital injection constraints are satisfied while minimizing a payoff function which is the sensitivity of a terminal constraint to a parameter variation. The unique feature of this problem is that the payoff function depends not only on the state variables and the controls, but also on the adjoint variables (Lagrange multipliers). In order to determine the influence of a control change on the payoff, one must take into account the change in the adjoint variables as well as the state variables. Linear perturbation analysis leads to two new sets of differential equations with associated boundary conditions. One set of equations is integrated forward along with the state equations while the other set is integrated backwards along with the adjoint equations.

Analysis

The gradient method of optimization is to be used to reduce error sensitivities. In order to apply this method, the linear perturbation equation relating changes in the control variables to changes in the sensitivity must first be obtained.

The equations of motion for a trajectory can be represented in the form

$$\dot{x} = f(x,\alpha,\xi,P,q) \tag{1}$$

The control and system variables are functions which influence the trajectory over a period of time, while the parameters affect the trajectory at only one time. Thrust attitude and the length of a coast between powered stages are examples of a control variable and a control parameter, respectively. The system variables and parameters are the quantities which cause trajectory errors when they have nonnominal values. The control variables, once they have been determined, become system variables because an error in the control is one of the major error sources. Thrust magnitude and atmospheric density are other examples of system variables and the burn time of a stage is an example of a system parameter.

In general, one will be interested in reducing the effect of all the significant error sources on all of the important terminal constraints. A discussion of this complete problem can be found in Ref. 3. Here, the analysis will be simplified by considering the effect of an error in only one system variable on one terminal constraint. This simplification reduces the complexity of the derivation without altering the essential features of the problem.

The linear perturbation equation relating changes in a system variable to changes in a terminal constraint can be written as

$$\delta\psi = \int_{t_i}^{t_f} \Lambda_P \delta P dt \tag{2}$$

where $\delta\psi$ is the change in the terminal constraint

$$\Lambda_P = \lambda_\psi{}^T(\partial f/\partial P) \tag{3}$$

λ_ψ is an n vector of adjoint variables which results from solving the set of differential equations

$$\dot{\lambda}_\psi = -(\partial f/\partial x)^T\lambda_\psi \tag{4}$$

with boundary conditions

$$\lambda_\psi{}^T(t_f) = (\partial\psi/\partial x)_{t=t_f} \tag{5}$$

A derivation of this relation can be found in Ref. 4.

The change in the system variable δP will be assumed constant. The sensitivity of the constraint ψ to the variable P is thus given by

$$\frac{d\psi}{dP} = \varphi = \int_{t_i}^{t_f} \Lambda_P dt \tag{6}$$

The quantity φ is the payoff to be minimized, or maximized, depending on its sign. φ is a function of x,α, and the adjoint variables associated with the constraint ψ. The unique feature of this payoff is the dependence on the adjoint variables. In order to determine the influence of the control on the payoff, one must take into account the change in the adjoint variables as well as the state variables.

The linear perturbation equation relating small changes in the sensitivity payoff to changes in the control has the same form as Eq. (2)

$$\delta\varphi = \int_{t_i}^{t_f} \Lambda_\varphi \delta\alpha dt \tag{7}$$

The purpose of the derivation that follows is to determine Λ_φ, taking into account that φ is a function of the adjoint variables. The approach is that of Ref. 5.

Form the quantity

$$\bar{F}(t,x,\dot{x},\lambda_\psi,\dot{\lambda}_\psi,\alpha) = \Lambda_P(x,\lambda_\psi,\alpha) + \nu^T(t)[f(x,\alpha) - \dot{x}] + \mu^T(t)[\dot{\lambda}_\psi + (\partial f/\partial x)^T\lambda_\psi] \tag{8}$$

$\nu(t)$ and $\mu(t)$ are n-vectors of adjoint variables or Lagrange multipliers. $\bar{F}$ is just the integrand of Eq. (6) so that

$$\varphi = \int_{t_i}^{t_f} \bar{F} dt \tag{9}$$

Now, proceeding formally, the differential of φ is given by

$$\delta\varphi = \int_{t_i}^{t_f} \left[\left(\frac{\partial\bar{F}}{\partial x}\right)\delta x + \left(\frac{\partial\bar{F}}{\partial\dot{x}}\right)\delta\dot{x} + \left(\frac{\partial\bar{F}}{\partial\lambda_\psi}\right)\delta\lambda_\psi + \left(\frac{\partial\bar{F}}{\partial\dot{\lambda}_\psi}\right)\delta\dot{\lambda}_\psi + \left(\frac{\partial\bar{F}}{\partial\alpha}\right)\delta\alpha\right]dt \tag{10}$$

The term involving $\delta\dot{x}$ is integrated by parts to give

$$\int_{t_i}^{t_f}\left(\frac{\partial\bar{F}}{\partial\dot{x}}\right)\delta\dot{x}dt = \left(\frac{\partial\bar{F}}{\partial\dot{x}}\right)\delta x\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f}\frac{d}{dt}\left(\frac{\partial\bar{F}}{\partial\dot{x}}\right)\delta x dt \tag{11}$$

Substituting for $\bar{F}$ from Eq. (8) yields

$$\int_{t_i}^{t_f}\left(\frac{\partial\bar{F}}{\partial\dot{x}}\right)\delta\dot{x}dt = \nu^{iT}\delta x^i - \nu^{fT}\delta x^f + \int_{t_i}^{t_f}\dot{\nu}^T\delta x dt \tag{12}$$

where the superscripts i and f indicate values at times t_i and

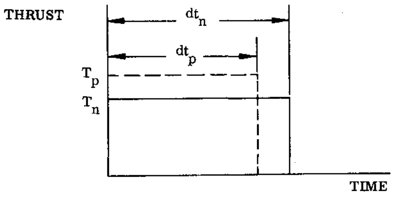

Fig. 1 Sketch of thrust histories.

t_f, respectively. Similarly the term involving $\delta\dot{\lambda}$ becomes

$$\int_{t_i}^{t_f} \frac{\partial \bar{F}}{\partial \dot{\lambda}_\psi} \delta\dot{\lambda}_\psi dt = \mu^{fT}\delta\lambda_\psi{}^f - \mu^{iT}\delta\lambda_\psi{}^i - \int_{t_i}^{t_f} \dot{\mu}^T\delta\lambda_\psi dt \quad (13)$$

After substituting Eqs. (12) and (13) in Eq. (10), one obtains

$$\delta\varphi = \nu^{iT}\delta x^i - \nu^{fT}\delta x^f + \mu^{fT}\delta\lambda_\psi{}^f - \mu^{iT}\delta\lambda_\psi{}^i + \int_{t_i}^{t_f}\left[\left(\frac{\partial \bar{F}}{\partial x} + \dot{\nu}^T\right)\delta x + \left(\frac{\partial \bar{F}}{\partial \lambda} - \dot{\mu}^T\right)\delta\lambda_\psi + \frac{\partial \bar{F}}{\partial \alpha}\delta\alpha\right]dt \quad (14)$$

ν and μ are selected so that the coefficients of δx and $\delta\lambda_\psi$ inside the integral are zero. This leads to the differential equations

$$\dot{\nu} = -(\partial\bar{F}/\partial x)^T \quad (15)$$

$$\dot{\mu} = (\partial\bar{F}/\partial\lambda_\psi)^T \quad (16)$$

The boundary conditions for these equations are chosen so that $\delta\varphi$ is not a function of unknown quantities. The change in the state at t_f, δx^f, is unknown. Therefore, its coefficient will be set to zero. This coefficient will involve μ^f. Note from Eq. (5) that

$$\delta\lambda_\psi(t_f) = (\partial/\partial x)(\partial\psi/\partial x)^T\delta x^f \quad (17)$$

Substituting Eq. (17) into Eq. (14) and setting the coefficient of δx^f to zero leads to the boundary condition for ν

$$\nu^{fT} = \mu^{fT}(\partial/\partial x)(\partial\psi/\partial x) \quad (18)$$

The adjoint variables λ_ψ are specified at the terminal time. The change in the variables at t_i is unknown. Therefore, the boundary condition for μ is chosen to be

$$u^{iT} = 0 \quad (19)$$

The functions ν and μ can be interpreted as sensitivities. $\nu(t)$ gives the change in φ due to a change in x at time t. This is identical to the relation between λ_ψ and ψ. $\mu(t)$ gives the change in φ due to a change in λ_ψ at time t.

If Eqs. (15, 16, 18, and 19) are substituted into Eq. (14) and δx^i is assumed to be zero, then Eq. (14) reduces to

$$\delta\varphi = \int_{t_i}^{t_f} \frac{\partial \bar{F}}{\partial \alpha}\delta\alpha dt \quad (20)$$

Comparing Eq. (20) with Eq. (7), it is seen that the desired influence coefficient Λ_φ is given by

$$\Lambda_\varphi = \partial\bar{F}/\partial\alpha = \partial\Lambda_P/\partial\alpha + \nu^T\partial f/\partial\alpha + \mu^T(\partial/\partial\alpha)[(\partial f/\partial x)^T\lambda_\psi] \quad (21)$$

How do the equations that have been derived here fit into the gradient method of optimization? The initial conditions for the equations involving $\dot{\mu}$ [Eq. (16)] are specified at t_i. These equations can, therefore, be integrated forward along with the state equations. The initial conditions for the $\dot{\nu}$ equations [Eq. (15)] are given at t_f. These equations which involve both μ and λ_ψ, are integrated backwards along with the usual adjoint equations [Eq. (4)]. Λ_φ, which is a function of ν,μ, and λ_ψ, is evaluated and stored along this backward run. Λ_φ is combined with the influence coefficients for the terminal constraints in the usual manner and an expression for $\delta\alpha$ which will reduce the payoff while meeting constraints is obtained. Thus, the basic sequence of forward and backward integration associated with the gradient method is maintained. The only difference is that several additional sets of differential equations must be integrated.

Numerical Example

The procedure described in the previous section will be used to minimize the sensitivity of terminal altitude to first stage burn rate for the Scout booster. The mission involves placing a payload into a reentry trajectory. There are terminal constraints on the altitude, flight-path angle, and downrange location of re-entry. There is also a constraint on dynamic pressure at the start of the second stage.

The Scout vehicle used for this mission has three stages. The vehicle coasts between the powered stages and the length of each of these coasts is a control parameter. Also, there is no requirement that the third stage burn out at the desired reentry point. Therefore, an adjustable coast is allowed after the third stage. The thrust orientation angle, measured with respect to an inertial coordinate system, is the control variable.

Assumptions

In order to reduce the programing complexity, the following simplifications have been made in the model: 1) motion is restricted to two dimensions; 2) an exponential atmosphere is used; 3) the lift coefficient is zero and the drag coefficient is independent of Mach number; 4) the thrust and mass flow rate in the first stage are constant.

Sensitivity Function

The payoff to be minimized is the sensitivity of terminal altitude to first stage burn rate. An increase in the burn rate implies an increase in the thrust and a decrease in the burn time such that the product of thrust and burn time remain constant. The burn rate variation will be represented as a thrust variation where it is understood that a one pound increase in thrust goes along with a decrease in burn time given by the ratio of the nominal burn time to the nominal thrust. Let the nominal stage time be broken up into segments of length dt_n. The thrust magnitude over each segment is T_n as shown in Fig. 1. When the thrust is increased to the perturbed value T_p, the length of each segment is reduced to dt_p.

The change in the terminal altitude, $\delta h(t_f)$, due to the perturbed thrust over the interval dt_n can be expressed as

$$\delta h(t_f) = \lambda^T d(\delta x) \quad (22)$$

where λ is the vector of adjoint variables associated with the altitude constraint and

$$d(\delta x) = x_p(dt_p) - x_n(dt_n) \quad (23)$$

i.e., the difference between the perturbed trajectory variables after time interval dt_p and the nominal variables after time dt_n.

The component of $d(\delta x)$ due to increased thrust is found by writing the differential of the trajectory equations (Eq. 1)

$$\delta\dot{x} = (\partial f/\partial T)\delta T \quad (24)$$

Over the time dt_n, the change in x is

$$d(\delta x_T) = (\partial f/\partial T)\delta T dt_n \quad (25)$$

The burn time is changed by

$$\delta t = -(dt_n/T_n)\delta T \quad (26)$$

If the rate of change of x is f, then $d(\delta x)$ due to the reduced burn time is

$$d(\delta x_t) = -(f/T_n)\delta T dt_n \quad (27)$$

The combined effect is found by adding Eqs. (25) and (27) to give

$$d(\delta x) = (\partial f/\partial T - f/T_n)\delta T dt_n \quad (28)$$

Substituting Eq. (28) into Eq. (22) yields

$$\delta h(t_f) = \lambda^T(\partial f/\partial T - f/T_n)\delta T dt_n \quad (29)$$

The change over an entire stage is found by integrating Eq. (29). The resulting sensitivity is

$$\frac{dh}{dT} = \int_{\text{stage 1}} \lambda^T \left(\frac{\partial f}{\partial T} - \frac{f}{T_n}\right) dt \quad (30)$$

The Λ_P of Eq. (6), then, is just

$$\Lambda_P = \lambda^T(\partial f/\partial T - f/T_n) \quad (31)$$

The sensitivity given by Eq. (30) is the payoff in the optimization procedure. In order to determine whether an improvement in payoff has been achieved on a forward run, Eq. (30) must be evaluated. Note, however, that it depends on the adjoint variables evaluated along that trajectory. These adjoint variables are not known until a backward integration of the trajectory has been made. Therefore, at the end of every forward run which meets terminal constraints, a special backward run is made to integrate the adjoint equations so that the payoff can be evaluated.

Variational Equations

The equations for μ are obtained by substituting Eq. (8) into Eq. (16). The result is

$$\dot{\mu} = (\partial f/\partial x)\mu + (\partial \Lambda_P/\partial \lambda)^T \quad (32)$$

From Eq. (19), the initial conditions are seen to be

$$\mu(t_i) = 0 \quad (33)$$

μ is a five-component vector with one component for each of the state variables, v,γ,r,τ,m. $\partial f/\partial x$ is a 5×5 matrix. The last term in Eq. (32) is added only during the first stage. From Eq. (31) it is

$$(\partial \Lambda_P/\partial \lambda)^T = (\partial f/\partial T - f/T_n) \quad (34)$$

From Eq. (15), the equations for ν are

$$= -(\partial \Lambda_P/\partial x)^T - (\partial f/\partial x)^T \nu - \{(\partial/\partial x)[(\partial f/\partial x)^T \lambda]\}^T \mu \quad (35)$$

with boundary conditions from Eq. (18) equal to

$$\nu(t_f) = 0 \quad (36)$$

ν is a five-component vector. The partial of Λ_P with respect to x is given by

$$\partial \Lambda_P/\partial x = \lambda^T[\partial^2 f/\partial T \partial x - (1/T_n)\partial f/\partial x] \quad (37)$$

The quantity in parentheses in the last term of Eq. (35) is the second-order partial of the Hamiltonian with respect to the state variables.

The influence coefficient relating the control θ to the sensitivity is given by substituting θ for α in Eq. (21). The result is

$$\Lambda_\varphi = \partial \Lambda_P/\partial \theta + \nu^T(\partial f/\partial \theta) + \mu^T(\partial/\partial \theta)[(\partial f/\partial x)^T \lambda] \quad (38)$$

The partial of Λ_P with respect to θ is given by

$$\partial \Lambda_P/\partial \theta = \lambda^T[\partial^2 f/\partial T \partial \theta - (1/T_n)\partial f/\partial \theta] \quad (39)$$

The final term in Eq. (38) can be written in the form

$$\mu^T(\partial/\partial \theta)[(\partial f/\partial x)^T \lambda] = \mu^T[\partial^2 f/\partial \theta \partial x]^T \lambda \quad (40)$$

The matrices required to evaluate Eqs. (32–40) are given in the Appendix.

Adjustable Coasts

The length of the coast after each powered stage is an adjustable parameter which must be optimized along with the thrust attitude history. What influence does a change in coast time have on the sensitivity? Referring to Eq. (14) and using the sensitivity interpretations for ν and μ, it is seen that the influence of perturbations in x and λ on φ is given by

$$\delta \varphi = \nu^T \delta x + \mu^T \delta \lambda \quad (41)$$

where the superscripts of Eq. (14) have been dropped. If the length of a coast is changed on a forward integration, a δx will appear at the end of the coast of magnitude[6]

$$\delta x = \dot{x}|_{t_{cf}} \delta t \quad (42)$$

On the other hand, λ is integrated backwards in time. A change in coast length leads to a $\delta\lambda$ at the beginning of the coast given by

$$\delta \lambda = -\dot{\lambda}|_{t_{ci}} \delta t \quad (43)$$

The combined effect is obtained by substituting Eqs. (42) and (43) into Eq. (41) to give

$$\delta \varphi = (\nu^T \dot{x}|_{t_{cf}} - \mu^T \dot{\lambda}|_{t_{ci}})\delta t \quad (44)$$

where t_{ci} and t_{cf} represent the times at the beginning and end of the coast. The quantity in parenthesis in Eq. (44) gives the effect of a change in coast time on the sensitivity payoff. This quantity is required to optimize the coast times along with the thrust attitude history as shown in Ref. 6.

Results

A number of cases were run to determine the reduction in sensitivity that can be obtained as the burnout weight is lowered. First, it was determined that the maximum burnout weight for the re-entry mission is 1392 lbs. The burnout weight was then fixed at lower values ranging down to 1200 lb and the sensitivity was minimized for each burnout weight. As the burnout weight is lowered, larger changes in the trajectory shape become possible and the sensitivity can be reduced to a greater degree. The results are shown in Fig. 2. It is seen that lowering the burnout weight from 1390 to 1200 lb makes a 50% reduction in sensitivity possible. The general trend of the trajectory shaping is to steepen the trajectory as shown in the altitude-range profiles in Fig. 3. The initial trajectory is similar to the maximum burnout weight trajectory. The control history for the maximum burnout case was used as the nominal control history for the 1200 lb case. The initial trajectory is the first trajectory in the sequence of iterations that satisfies all the terminal constraints. It should be noted that the sensitivity for the initial trajectory is 42.3, which is almost as high as the sensitivity for the maximum weight case. This indicates that the reduction iin sensitivity is due to shaping the trajectory and not merely to lowering the burnout weight.

The sensitivities plotted in Fig. 2 come from evaluating the integral in Eq. (30). To check this integral, the burn rate was perturbed for the two trajectories in Fig. 3 and the change in terminal altitude was observed. A comparison of the integral and the perturbation evaluation of sensitivity is shown in Table 1. The agreement between the two calculations is good.

It is interesting to note that the trajectory that minimizes the altitude sensitivity also reduces the sensitivity of the other terminal constraints to burn rate changes. The effect of a 400-lb change in thrust on the terminal constraints for the two trajectories in Fig. 3 is shown in Table 2.

Conclusions

There are two conclusions to be drawn from this study: 1) trajectory shaping can significantly reduce the sensitivity

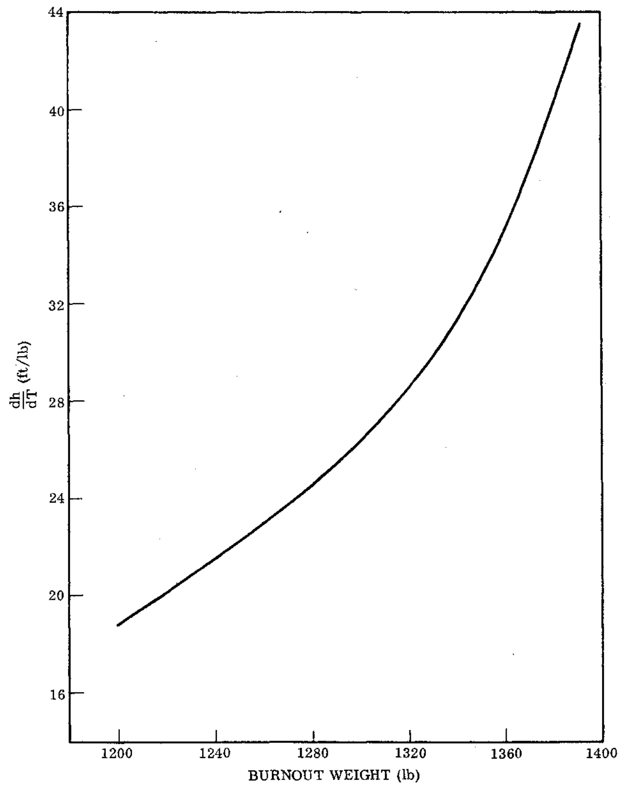

Fig. 2 Minimum sensitivity as a function of burnout weight.

of terminal constraints to variations in system parameters; and 2) the gradient method of optimization can be used to find the minimum sensitivity trajectory. In view of the success achieved in minimizing the sensitivity to one error source, it would seem fruitful to apply this method to a problem in which all the major error sources are included simultaneously.

Appendix: Equations of Motion and Matrices

The equations of motion in the coordinate system shown in Fig. 4 are

$$F = \dot{v} = -g\sin\gamma + (T/m)(C_{22}\cos\theta + C_{23}\sin\theta) - D/m$$

$$G = \dot{\gamma} = (v/r - g/v)\cos\gamma + (T/mv)(C_{32}\cos\theta + C_{33}\sin\theta)$$

$$I = \dot{r} = v\sin\gamma,\ K = \dot{\tau} = v\cos\gamma/r,\ N = \dot{m} = -T/g_0 I_{sp}$$

where

$$C_{22} = \cos\tau\cos\gamma + \sin\tau\sin\gamma,\ C_{33} = C_{22}$$

$$C_{23} = -\sin\tau\cos\gamma + \cos\tau\sin\gamma,\ C_{32} = -C_{23}$$

$$D = \tfrac{1}{2}\rho v^2 C_D A,\ C_D \text{ independent of Mach number}$$

$$\rho = \rho_0 e^{-\beta h},\ \beta = 1/23000$$

f is a column vector given by

$$f^T = [F,G,I,K,N]$$

The partial derivative matrix of f with respect to x is

$$\partial f/\partial x = \begin{bmatrix} \frac{\partial F}{\partial v} & \frac{\partial F}{\partial \gamma} & \frac{\partial F}{\partial r} & \frac{\partial F}{\partial \tau} & \frac{\partial F}{\partial m} \\ \frac{\partial G}{\partial v} & \frac{\partial G}{\partial \gamma} & \frac{\partial G}{\partial r} & \frac{\partial G}{\partial \tau} & \frac{\partial G}{\partial m} \\ \frac{\partial I}{\partial v} & \frac{\partial I}{\partial \gamma} & 0 & 0 & 0 \\ \frac{\partial K}{\partial v} & \frac{\partial K}{\partial \gamma} & \frac{\partial K}{\partial r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The partial derivatives of f with respect to T are

$$(\partial f/\partial T)^T = [(\partial F/\partial T)(\partial G/\partial T)\ 0\ 0\ (\partial N/\partial T)]$$

The second-order partial of f with respect to the vector x and the scalar T is found by taking the partial of each term in $\partial f/\partial x$ with respect to T.

$$\partial^2 f/\partial T\partial x = \begin{bmatrix} 0 & F_{T\gamma} & 0 & F_{T\tau} & F_{Tm} \\ G_{Tv} & G_{T\gamma} & 0 & G_{T\tau} & G_{Tm} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The second-order partial of the Hamiltonian is

$$H_{xx} = (\partial/\partial x)[(\partial f/\partial x)^T\lambda]$$

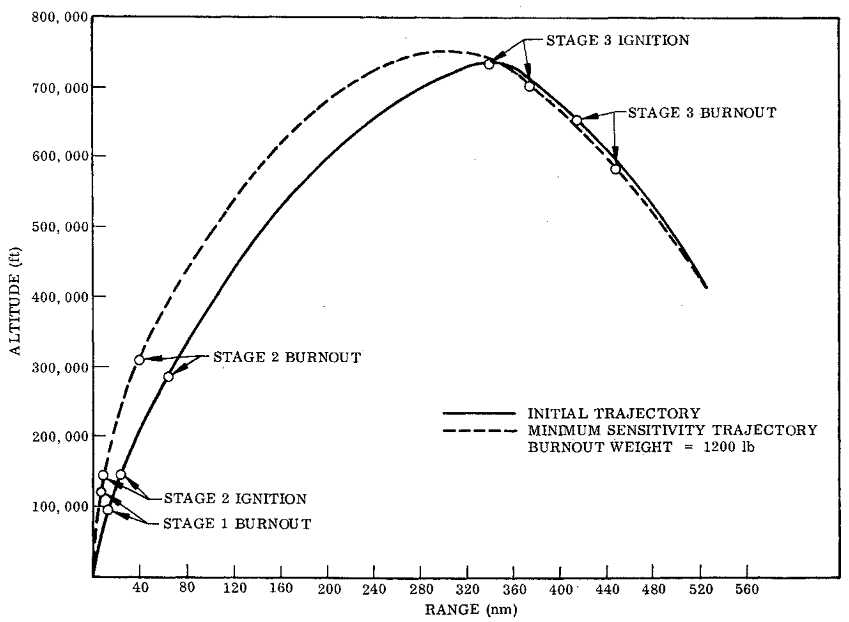

Fig. 3 Comparison of initial and minimum sensitivity trajectories.

Table 1 Sensitivity evaluation

	Integral	Perturbations
Initial trajectory	42.3	43.2
Minimum sensitivity trajectory	18.9	17.3

The term in brackets is a column vector. Each component of the vector is converted to a row when the partial with respect to the state vector is taken. The components of H_{xx} can be written as

$$\begin{bmatrix} H_{vv} \\ H_{v\gamma} \\ H_{vr} \\ H_{v\tau} \\ H_{vm} \end{bmatrix} = \begin{bmatrix} F_{vv} & G_{vv} & 0 & 0 & 0 \\ 0 & G_{v\gamma} & I_{v\gamma} & K_{v\gamma} & 0 \\ F_{vr} & G_{vr} & 0 & K_{vr} & 0 \\ 0 & G_{v\tau} & 0 & 0 & 0 \\ F_{vm} & G_{vm} & 0 & 0 & 0 \end{bmatrix} [\lambda]$$

$$\begin{bmatrix} H_{\gamma\gamma} \\ H_{\gamma r} \\ H_{\gamma\tau} \\ H_{\gamma m} \end{bmatrix} = \begin{bmatrix} F_{\gamma\gamma} & G_{\gamma\gamma} & I_{\gamma\gamma} & K_{\gamma\gamma} & 0 \\ F_{\gamma r} & G_{\gamma r} & 0 & K_{\gamma r} & 0 \\ F_{\gamma\tau} & G_{\gamma\tau} & 0 & 0 & 0 \\ F_{\gamma m} & G_{\gamma m} & 0 & 0 & 0 \end{bmatrix} [\lambda]$$

$$\begin{bmatrix} H_{rr} \\ H_{r\tau} \\ H_{rm} \end{bmatrix} = \begin{bmatrix} F_{rr} & G_{rr} & 0 & K_{rr} & 0 \\ F_{r\tau} & G_{r\tau} & 0 & 0 & 0 \\ F_{rm} & G_{rm} & 0 & 0 & 0 \end{bmatrix} [\lambda]$$

$$\begin{bmatrix} H_{\tau\tau} \\ H_{\tau m} \end{bmatrix} = \begin{bmatrix} F_{\tau\tau} & G_{\tau\tau} & 0 & 0 & 0 \\ F_{\tau m} & G_{\tau m} & 0 & 0 & 0 \end{bmatrix} [\lambda]$$

$$[H_{mm}] = [F_{mm} \quad G_{mm} \quad 0 \quad 0 \quad 0] \, [\lambda]$$

The partial of f with respect to θ is

$$(\partial f/\partial\theta)^T = [(\partial F/\partial\theta)(\partial G/\partial\theta)\ 0\ 0\ 0]$$

Table 2 Perturbations in terminal constraints

Constraint	Initial trajectory	Minimum sensitivity trajectory
Altitude, ft	17276	6901
Flight path angle, deg	0.08	0.02
Velocity, fps	−59.6	−28.3
Range, naut miles	−2.3	−1.5

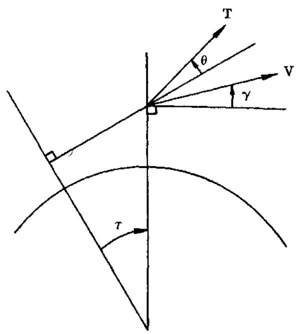

Fig. 4 Coordinate system for equations of motion.

The second-order partial of f with respect to the scalars T and θ is

$$\partial^2 f/\partial T\partial\theta = [(\partial^2 F/\partial T\partial\theta)(\partial^2 G/\partial T\partial\theta)\ 0\ 0\ 0]$$

The second-order partial of f with respect to x and θ is found by taking the derivative of each component of $\partial f/\partial x$ with respect to θ

$$\partial^2 f/\partial\theta\partial x = \begin{bmatrix} 0 & F_{\theta\gamma} & F_{\theta r} & F_{\theta\tau} & F_{\theta m} \\ G_{\theta v} & G_{\theta\gamma} & G_{\theta r} & G_{\theta\tau} & G_{\theta m} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The components of these matrices are given in Ref. 3.

References

[1] Leondes, C. T. and Volgenau, E., "Improvement of Missile and Space Vehicle Accuracy by Trajectory Optimization," *Journal of Spacecraft and Rockets*, Vol. 4, No. 12, Dec. 1967, pp. 1609–1612.

[2] Watson, J. W. and Stubberud, A. R., "Atmospheric Entry Employing Range Sensitivity," *Journal of Spacecraft and Rockets*, Vol. 5, No. 8, Aug. 1968, pp. 983–984.

[3] Rosenbaum, R. and Willwerth, R. E., "Launch Vehicle Error Sensitivity Study," CR-1512, Feb. 1970, NASA.

[4] Bryson, A. E. and Ho, Y. C., *Applied Optimum Control*, Blaisdell, Waltham, Mass., 1969, pp. 47–49.

[5] Breakwell, J. V., "The Optimization of Trajectories," *SIAM Journal*, Vol. 7, No. 2, June 1959, pp. 215–247.

[6] Rosenbaum, R., Willwerth, R. E., and Chuck, W., "Powered Flight Trajectory Optimization for Lunar and Interplanetary Flight," *Astronautica Acta*, Vol. 12, No. 2, April 1966, pp. 159–168.